PHYS 705: Classical Mechanics

Canonical Transformation

Canonical Variables and Hamiltonian Formalism

As we have seen, in the Hamiltonian Formulation of Mechanics,

→ q_j, p_j are independent variables in phase space on equal footing

→ The Hamilton's Equation for q_j, p_j are "symmetric" (*symplectic*, later)

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \quad \mathit{and} \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

→ This elegant formal structure of mechanics affords us the freedom in selecting other (may be better) canonical variables as our phase space "coordinates" and "momenta"

- As long as the new variables formally satisfy this abstract structure (the form of the Hamilton's Equations.

Canonical Transformation

Recall (from hw) that the Euler-Lagrange Equation is invariant for a point transformation: $Q_j = Q_j(q,t)$

i.e., if we have,
$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0,$$

then,
$$\frac{\partial L}{\partial Q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_j}\right) = 0,$$

Now, the idea is to find a generalized (canonical) transformation in *phase space* (not config. space) such that the Hamilton's Equations are invariant !

$$Q_j = Q_j(q,p,t)$$
$$P_j = P_j(q,p,t)$$

(In general, we look for transformations which are *invertible*.)

Invariance of EL equation for Point Transformation

First look at the situation in config. space first:

Given: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0,$$ and a point transformation: $Q_j = Q_j(q,t)$

→ Need to show: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_j}\right)-\frac{\partial L}{\partial Q_j}=0$$

Formally, calculate: $$\frac{\partial L}{\partial Q_j}=\sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial Q_j}+\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial Q_j}$$ (chain rule)

$$\frac{\partial L}{\partial \dot{Q}_j}=\sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \dot{Q}_j}+\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \dot{Q}_j}$$

From the inverse point transformation equation $q_i = q_i(Q,t)$, we have,

$$\frac{\partial q_i}{\partial \dot{Q}_j}=0$$ and $$\frac{\partial \dot{q}_i}{\partial \dot{Q}_j}=\frac{\partial q_i}{\partial Q_j}$$

$$\boxed{\dot{q}_i=\sum_k \frac{\partial q_i}{\partial Q_k}\dot{Q}_k+\frac{\partial q_i}{\partial t}}$$

Invariance of EL equation for Point Transformation

Forming the LHS of EL equation with Q_j: $LHS = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_j}\right) - \frac{\partial L}{\partial Q_j}$

$$LHS = \frac{d}{dt}\left\{\sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial Q_j}\right\} - \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial Q_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial Q_j}$$

$$= \sum_i \left\{\frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\frac{\partial q_i}{\partial Q_j}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial Q_j}\right\} - \sum_i \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial Q_j} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial Q_j}$$

$$= \sum_i \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial Q_j} - \left(\frac{\partial L}{\partial q_i}\right)\frac{\partial q_i}{\partial Q_j} + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\left(\frac{\partial q_i}{\partial Q_j}\right) - \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial Q_j}\right\}$$

Invariance of EL equation for Point Transformation

$$LHS = \sum_i \left\{ \underbrace{\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) \right]}_{\substack{=0 \\ \text{(Since that's what given !)}}} \frac{\partial q_i}{\partial Q_j} + \frac{\partial L}{\partial \dot{q}_i} \underbrace{\left[\overbrace{\frac{d}{dt}\left(\frac{\partial q_i}{\partial Q_j}\right)}^{\text{(exchange order of diff)}} - \frac{\partial \dot{q}_i}{\partial Q_j} \right]} \right\}$$

$$= \frac{\partial}{\partial Q_j}\frac{dq_i}{dt} - \frac{\partial \dot{q}_i}{\partial Q_j}$$

$$= \frac{\partial \dot{q}_i}{\partial Q_j} - \frac{\partial \dot{q}_i}{\partial Q_j} = 0$$

$$LHS = 0 \quad \Rightarrow \quad \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_j}\right) - \frac{\partial L}{\partial Q_j} = 0}$$

Canonical Transformation

Now, back to phase space with q's & p's, we need to find the appropriate (canonical) transformation $Q_j = Q_j(q, p, t)$ and $P_j = P_j(q, p, t)$ such that there exist a transformed Hamiltonian $K(Q, P, t)$ with which the Hamilton's Equations are satisfied:

$$\dot{Q}_j = \frac{\partial K}{\partial P_j} \quad and \quad \dot{P}_j = -\frac{\partial K}{\partial Q_j}$$

(The form of the EOM must be *invariant* in the new coordinates.)

** It is important to further stated that the transformation considered must also be *problem-independent* meaning that (Q, P) must be canonical coordinates for all system with the same number of dofs.

Canonical Transformation

To see what this condition might say about our canonical transformation, we need to go back to the Hamilton's Principle:

Hamilton's Principle: The motion of the system in *configuration space* is such that the action I has a stationary value for the actual path, .i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L dt = 0$$

Now, we need to extend this to the 2*n*-dimensional *phase space*

1. The integrant in the action integral must now be a function of the independent conjugate variable q_j, p_j and their derivatives $\dot{q}_j, \dot{p}_j$
2. We will consider variations in all 2*n* phase space coordinates

Hamilton's Principle in Phase Space

1. To rewrite the integrant in terms of $q_j, p_j, \dot{q}_j, \dot{p}_j$, we will utilize the definition for the Hamiltonian (or the inverse Legendre Transform):

$$H = p_j \dot{q}_j - L \quad \rightarrow \quad L = p_j \dot{q}_j - H(q,p,t) \qquad \text{(Einstein's sum rule)}$$

Substituting this into our variation equation, we have

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} \left[p_j \dot{q}_j - H(q,p,t) \right] dt = 0$$

2. The variations are now for $n\, q_j$'s and $n\, p_j$'s : (all q's and p's are *independent*)

The rewritten integrant $\Gamma(q, \dot{q}, p) = p_j \dot{q}_j - H(q,p,t)$ is formally a function of $q_j, p_j, \dot{q}_j, \dot{p}_j$ but in fact it does not depend on $\dot{p}_j$, i.e. $\partial\Gamma / \partial \dot{p}_j = 0$ This fact will proved to be useful later on.

→again, we will required the variations for the q_j to be zero at ends

Hamilton's Principle in Phase Space

$$\boxed{\begin{aligned} q_j &= q_{0j} + \alpha\eta_j \\ p_j &= p_{0j} + \alpha\mu_j \end{aligned}}$$

Affecting the variations on all $2n$ variables q_j, p_j , we have,

$$\frac{\partial I}{\partial \alpha}d\alpha = \int_{t_1}^{t_2}\left\{\sum_j\left(\frac{\partial \Gamma}{\partial q_j}\frac{\partial q_j}{\partial \alpha}d\alpha + \frac{\partial \Gamma}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \alpha}d\alpha\right)\right. \quad \Leftarrow \quad \delta q_j\text{'s}$$

$$\delta p_j\text{'s} \quad \Rightarrow \quad \left. + \sum_j\left(\frac{\partial \Gamma}{\partial p_j}\frac{\partial p_j}{\partial \alpha}d\alpha + \frac{\partial \Gamma}{\partial \dot{p}_j}\frac{\partial \dot{p}_j}{\partial \alpha}d\alpha\right)\right\}dt = 0$$

As in previous discussion, the second term in the sum for δq_j's can be rewritten using integration by parts:

$$\int_{t_1}^{t_2}\frac{\partial \Gamma}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \alpha}dt = \left.\frac{\partial \Gamma}{\partial \dot{q}_j}\frac{\partial q_j}{\partial \alpha}\right|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\partial q_j}{\partial \alpha}\frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}_j}\right)dt$$

Hamilton's Principle in Phase Space

→ Previously, we have required the variations for the q_j to be zero at end pts

so that, $\left.\dfrac{\partial q_j}{\partial \alpha}\right|_{t=t_1,t_2} = 0 \quad \Rightarrow \quad \displaystyle\int_{t_1}^{t_2} \frac{\partial \Gamma}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \alpha} dt = \cancel{\left.\frac{\partial \Gamma}{\partial \dot{q}_j}\frac{\partial q_j}{\partial \alpha}\right|_{t_1}^{t_2}} - \int_{t_1}^{t_2} \frac{\partial q_j}{\partial \alpha}\frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}_j}\right) dt$

So, the first sum with δq_j's can be written as:

$$\int_{t_1}^{t_2} \sum_j \left(\frac{\partial \Gamma}{\partial q_j}\frac{\partial q_j}{\partial \alpha} d\alpha - \frac{\partial q_j}{\partial \alpha}\frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}_j}\right) d\alpha \right) dt$$

$$= \int_{t_1}^{t_2} \sum_j \left[\frac{\partial \Gamma}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}_j}\right) \right] \delta q_j dt \quad \text{where } \delta q_j = \frac{\partial q_j}{\partial \alpha} d\alpha$$

Hamilton's Principle in Phase Space

Now, perform the same integration by parts to the corresponding term for δp_j's

we have, $$\int_{t_1}^{t_2} \frac{\partial \Gamma}{\partial \dot{p}_j}\frac{\partial \dot{p}_j}{\partial \alpha}dt = \left.\frac{\partial \Gamma}{\partial \dot{p}_j}\frac{\partial p_j}{\partial \alpha}\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial p_j}{\partial \alpha}\frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{p}_j}\right)dt$$

→ Note, since $\partial\Gamma / \partial \dot{p}_j = 0$, $\left.\frac{\partial \Gamma}{\partial \dot{p}_j}\frac{\partial p_j}{\partial \alpha}\right|_{t_1}^{t_2} = 0$ without enforcing the variations for p_j to be zero at end points.

This gives the result for the 2nd sum in the variation equation for δp_j's :

$$\int_{t_1}^{t_2} \sum_j \left[\frac{\partial \Gamma}{\partial p_j} - \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{p}_j}\right)\right]\delta p_j dt \quad \text{where } \delta p_j = \frac{\partial p_j}{\partial \alpha}d\alpha$$

Hamilton's Principle in Phase Space

Putting both terms back together, we have:

$$\frac{\partial I}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \left\{ \sum_j \left[\frac{\partial \Gamma}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}_j} \right) \right] \delta q_j + \sum_j \left[\frac{\partial \Gamma}{\partial p_j} - \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{p}_j} \right) \right] \delta p_j \right\} dt = 0$$

① ②

Since both variations are independent, ① and ② must vanish *independently* !

① $\Rightarrow$ $$\frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}_j} \right) - \frac{\partial \Gamma}{\partial q_j} = 0$$

$$\Gamma(q, \dot{q}, p) = p_j \dot{q}_j - H(q, p, t)$$

$$\Rightarrow \frac{\partial \Gamma}{\partial \dot{q}_j} = p_j \quad \text{and} \quad \frac{\partial \Gamma}{\partial q_j} = -\frac{\partial H}{\partial q_j}$$

$$\dot{p}_j - \left[-\frac{\partial H}{\partial q_j} \right] = 0$$

$\Rightarrow$ $$\dot{p}_j = -\frac{\partial H}{\partial q_j}$$

(one of the Hamilton's equations)

Hamilton's Principle in Phase Space

$$\frac{\partial I}{\partial \alpha} d\alpha = \int_{t_1}^{t_2} \left\{ \sum_j \left[\frac{\partial \Gamma}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{q}_j} \right) \right] \delta q_j + \sum_j \left[\frac{\partial \Gamma}{\partial p_j} - \frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{p}_j} \right) \right] \delta p_j \right\} dt = 0$$

①　②

② $\Rightarrow$ $$\frac{d}{dt}\left(\frac{\partial \Gamma}{\partial \dot{p}_j} \right) - \frac{\partial \Gamma}{\partial p_j} = 0$$

$$\Gamma(q, \dot{q}, p) = p_j \dot{q}_j - H(q, p, t)$$

$$\Rightarrow \frac{\partial \Gamma}{\partial \dot{p}_j} = 0 \quad \text{and} \quad \frac{\partial \Gamma}{\partial p_j} = \dot{q}_j - \frac{\partial H}{\partial p_j}$$

$$0 - \left[\dot{q}_j - \frac{\partial H}{\partial p_j} \right] = 0$$

$\Rightarrow$ $$\dot{q}_j = \frac{\partial H}{\partial p_j}$$

(2nd Hamilton's equations)

Hamilton’s Principle in Phase Space

So, we have just shown that applying the Hamilton’s Principle in Phase Space, the resulting dynamical equation is the Hamilton’s Equations.

$$\begin{cases} \dot{p}_j = -\dfrac{\partial H}{\partial q_j} \\[2ex] \dot{q}_j = \dfrac{\partial H}{\partial p_j} \end{cases}$$

Hamilton's Principle in Phase Space

Notice that a *full* time derivative of an arbitrary function F of (q,p,t) can be put into the integrand of the action integral without affecting the variations:

$$\int_{t_1}^{t_2}\left[p_j\dot{q}_j - H(q,p,t) + \frac{dF}{dt}\right]dt$$

$$= \int_{t_1}^{t_2}\left[p_j\dot{q}_j - H(q,p,t)\right]dt + \underbrace{\int_{t_1}^{t_2}\frac{dF}{dt}dt}$$

$$= \int_{t_1}^{t_2} dF = F\Big|_{t_1}^{t_2} = const$$

Thus, when variation is taken, this constant term will not contribute !

Canonical Transformation

Now , we come back to the question: When is a transformation to Q, P canonical?

→ We need Hamilton's Equations to hold in both systems

This means that we need to have the following variational conditions:

$$\delta\int\left[p_j\dot{q}_j - H(q,p,t)\right]dt = 0 \quad \text{AND} \quad \delta\int\left[P_j\dot{Q}_j - K(Q,P,t)\right]dt = 0$$

→ For this to be true simultaneously, the integrands must equal

→ And, from our previous slide, this is also true if they are differed by a full time derivative of a function of *any* of the phase space variables involved + time:

$$p_j\dot{q}_j - H(q,p,t) = P_j\dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}\left(q,p,Q,P,t\right)$$

Canonical Transformation

$$p_j \dot{q}_j - H(q,p,t) = P_j \dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}\left(q,p,Q,P,t\right) \quad (*)\ (G9.11)$$

F is called the *Generating Function* for the canonical transformation:

$$(q_j, p_j) \to \left(Q_j, P_j\right): \begin{cases} Q_j = Q_j(q,p,t) \\ P_j = P_j(q,p,t) \end{cases}$$

→ As the name implies, different choice of *F* give us the ability to generate different Canonical Transformation to get to different $\left(Q_j, P_j\right)$

→ *F* is useful in specifying the exact form of the transformation if it contains half of the *old* variables and half of the *new* variables. It, then, acts as a bridge between the two sets of canonical variables.

Canonical Transformation

$$p_j\dot{q}_j - H(q,p,t) = P_j\dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}\left(q,p,Q,P,t\right) \quad (*)\ (G9.11)$$

F is called the *Generating Function* for the canonical transformation:

$$(q_j, p_j) \rightarrow \left(Q_j, P_j\right): \begin{cases} Q_j = Q_j(q,p,t) \\ P_j = P_j(q,p,t) \end{cases}$$

→ Depending on the form of the generating functions (which pair of canonical variables being considered as the *independent* variables for the Generating Function), we can classify canonical transformations into four basic types.

Canonical Transformation: 4 Types

$$p_j \dot{q}_j - H(q,p,t) = P_j \dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}\left(old, new, t\right)$$

Type 1: $F = F_1(q,Q,t)$	$p_j = \frac{\partial F_1}{\partial q_j}(q,Q,t)$	$P_j = -\frac{\partial F_1}{\partial Q_j}(q,Q,t)$	$K = H + \frac{\partial F_1}{\partial t}$
Type 2: $F = F_2(q,P,t) - Q_i P_i$	$p_j = \frac{\partial F_2}{\partial q_j}(q,P,t)$	$Q_j = \frac{\partial F_2}{\partial P_j}(q,P,t)$	$K = H + \frac{\partial F_2}{\partial t}$
Type 3: $F = F_3(p,Q,t) + q_i p_i$	$q_j = -\frac{\partial F_3}{\partial p_j}(p,Q,t)$	$P_j = -\frac{\partial F_3}{\partial Q_j}(p,Q,t)$	$K = H + \frac{\partial F_3}{\partial t}$
Type 4: $F = F_4(p,P,t) + q_i p_i - Q_i P_i$	$q_j = -\frac{\partial F_4}{\partial p_j}(p,P,t)$	$Q_j = \frac{\partial F_4}{\partial P_j}(p,P,t)$	$K = H + \frac{\partial F_4}{\partial t}$

Canonical Transformation: Type 1

(old) (new)

Type 1: $F = F_1(q, Q, t)$ | F is a function of q and Q + time

Writing out the full time derivative for F, Eq (1) becomes:

$$p_j \dot{q}_j - H = P_j \dot{Q}_j - K + \overbrace{\frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_j}\dot{q}_j + \frac{\partial F_1}{\partial Q_j}\dot{Q}_j}$$

(again E's sum rule)

Or, we can write the equation in differential form:

(write out $\dot{q}_j = \dfrac{dq_j}{dt}$, and $\dot{Q}_j = \dfrac{dQ_j}{dt}$ and multiply the equation by dt)

$$\Rightarrow \quad \left(p_j - \frac{\partial F_1}{\partial q_j}\right) dq_j - \left(P_j + \frac{\partial F_1}{\partial Q_j}\right) dQ_j + \left(K - H - \frac{\partial F_1}{\partial t}\right) dt = 0$$

Canonical Transformation: Type 1

Since all the q_j, Q_j are independent, their coefficients must vanish independently. This gives the following set of equations:

$$p_j = \frac{\partial F_1}{\partial q_j}(q,Q,t) \quad (C1) \qquad P_j = -\frac{\partial F_1}{\partial Q_j}(q,Q,t) \quad (C2) \qquad K = H + \frac{\partial F_1}{\partial t} \quad (C3)$$

These are the equations in the Table 9.1 in the book.

For a given specific expression for $F_1(q,Q,t)$, e.g. $F_1(q,Q,t) = q_j Q_j$

$\rightarrow Eq.\,(C1)$ are n relations defining p_j in terms of q_j, Q_j, t and they can be inverted to get the 1st set of the canonical transformation:

In the specific example, we have:

$$p_j = \frac{\partial F_1}{\partial q_j}(q,Q,t) = Q_j \quad \rightarrow \quad \boxed{Q_j = p_j}$$

Canonical Transformation: Type 1

$\rightarrow Eq.(C2)$ are n relations defining P_j in terms of q_j, Q_j, t . Together with our results for the Q_j, the 2nd set of the canonical transformation can be obtained.

Again, in the specific example, we have:

$$P_j = -\frac{\partial F_1}{\partial Q_j}(q, Q, t) = -q_j \quad \rightarrow \quad \boxed{P_j = -q_j}$$

$\rightarrow Eq.(C3)$ gives the connection between K and H:

$$K = H + \cancel{\frac{\partial F_1}{\partial t}} \quad \rightarrow \quad \boxed{K = H}$$

(note: $K(Q, P, t)$ is a function of the new variables so that the RHS needs to be *re-express* in terms of Q_j, P_j using the canonical transformation.)

Canonical Transformation: Type 1

In summary, for the specific example of a Type 1 generating function:

$$F_1(q,Q,t) = q_j Q_j$$

We have the following:

$$\left.\begin{aligned} Q_j &= p_j \\ P_j &= -q_j \end{aligned}\right\} \quad \text{Canonical Transformation}$$

and $\quad K = H \quad$ Transformed Hamiltonian

Note: this example results in basically swapping the generalized coordinates with their conjugate momenta in their dynamical role and this exercise demonstrates that swapping them basically results in the same situation !

→ Emphasizing the equal role for *q* and *p* in Hamiltonian Formalism !

Canonical Transformation: Type 2

(old) (new)

Type 2: $F = F_2(q, P, t) - Q_j P_j$, where F_2 is a function of q and P + time

(One can think of F_2 as **the Legendre transform** of $F(q,Q,t)$ in exchanging the variables Q and P.)

Substituting into our defining equation for canonical transformation, Eq. (1):

$$p_j \dot{q}_j - H = \cancel{P_j \dot{Q}_j} - K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_j}\dot{q}_j + \frac{\partial F_2}{\partial P_j}\dot{P}_j - \cancel{P_j \dot{Q}_j} - \dot{P}_j Q_j$$

Again, writing the equation in differential form:

$$\left(p_j - \frac{\partial F_2}{\partial q_j}\right) dq_j + \left(Q_j - \frac{\partial F_2}{\partial P_j}\right) dP_j + \left(K - H - \frac{\partial F_2}{\partial t}\right) dt = 0$$

Canonical Transformation: Type 2

Since all the q_j, P_j are independent, their coefficients must vanish independently. This gives the following set of equations:

$$p_j = \frac{\partial F_2}{\partial q_j}(q, P, t) \qquad Q_j = \frac{\partial F_2}{\partial P_j}(q, P, t) \qquad K = H + \frac{\partial F_2}{\partial t}$$

For a given specific expression for $F_2(q, P, t)$, e.g. $F_2(q, P, t) = q_j P_j$

$$p_j = \frac{\partial F_2}{\partial q_j}(q, P, t) = P_j \quad \rightarrow \quad \boxed{P_j = p_j}$$

$$Q_j = \frac{\partial F_2}{\partial P_j}(q, P, t) = q_j \quad \rightarrow \quad \boxed{Q_j = q_j}$$

$$\boxed{K = H}$$

Thus, the identity transformation is also a canonical transformation !

Canonical Transformation: Type 2

Let consider a slightly more general example for type 2: $F = F_2\left(q, P, t\right) - Q_j P_j$

with $F_2\left(q, P, t\right) = f\left(q_1, \cdots, q_n, t\right) P_j + g\left(q_1, \cdots, q_n, t\right)$

where f and g are function of q's only + time

Going through the same procedure, we will get:

$$p_j = \frac{\partial F_2}{\partial q_j}$$

$$\rightarrow \quad p_j = \frac{\partial f}{\partial q_j} P_j + \frac{\partial g}{\partial q_j}$$

$$Q_j = \frac{\partial F_2}{\partial P_j}$$

$$\rightarrow \quad \boxed{Q_j = f\left(q_1, \cdots q_n, t\right)}$$

$$K = H + \frac{\partial f}{\partial t} P_j + \frac{\partial g}{\partial t}$$

Notice that the Q equation is the general point transformation in the configuration space. In order for this to be canonical, the P and H transformations must be handled carefully (not necessary simple functions).

Canonical Transformation: Summary

The remaining two basic types are Legendre transformation of the remaining two variables:

$$F = F_3(p, Q, t) + p_j q_j \qquad q \leftrightarrow p$$

$$F = F_4(p, P, t) + p_j q_j - Q_j P_j \qquad q \leftrightarrow p \;\&\; Q \leftrightarrow P$$

(Results are summarized in Table 9.1 on p. 373 in Goldstein .)

Canonical Transformations form a group with the following properties:

1. The identity transformation is canonical (type 2 example)
2. If a transformation is canonical, so is its inverse
3. Two successive canonical transformations ("product") is canonical
4. The product operation is associative

$$Q_j = Q_j(q,p,t)$$
$$P_j = P_j(q,p,t)$$

Canonical Transformation: 4 Types

$$p_j \dot{q}_j - H(q,p,t) = P_j \dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}(old, new, t)$$

dep var

Type 1:
$F = F_1(q,Q,t)$

$$p_j = \frac{\partial F_1}{\partial q_j}(q,Q,t) \quad P_j = -\frac{\partial F_1}{\partial Q_j}(q,Q,t) \quad K = H + \frac{\partial F_1}{\partial t}$$

ind var

Type 2:
$F = F_2(q,P,t) - Q_i P_i$

$$p_j = \frac{\partial F_2}{\partial q_j}(q,P,t) \quad Q_j = \frac{\partial F_2}{\partial P_j}(q,P,t) \quad K = H + \frac{\partial F_2}{\partial t}$$

Type 3:
$F = F_3(p,Q,t) + q_i p_i$

$$q_j = -\frac{\partial F_3}{\partial p_j}(p,Q,t) \quad P_j = -\frac{\partial F_3}{\partial Q_j}(p,Q,t) \quad K = H + \frac{\partial F_3}{\partial t}$$

Type 4:
$F = F_4(p,P,t) + q_i p_i - Q_i P_i$

$$q_j = -\frac{\partial F_4}{\partial p_j}(p,P,t) \quad Q_j = \frac{\partial F_4}{\partial P_j}(p,P,t) \quad K = H + \frac{\partial F_4}{\partial t}$$

Canonical Transformation (more)

If we are given a canonical transformation

$$\begin{aligned} Q_j &= Q_j(q, p, t) \\ P_j &= P_j(q, p, t) \end{aligned} \qquad (*)$$

How do we find the appropriate generating function F ?

- Let say, we wish to find a generating function of the 1st type, i.e., $F = F_1(q, Q, t)$

(Note: generating function of the other types can be obtain through an appropriate Legendre transformation.)

- Since our chosen generating function (1st type) depends on q, Q, and t explicitly, we will rewrite our p and P in terms of q and Q using Eq. (*):

$$p_j = p_j(q, Q, t) \qquad\qquad P_j = P_j(q, Q, t)$$

Canonical Transformation (more)

Now, from the pair of equations for the Generating Function Derivatives (Table 9.1), we form the following diff eqs,

$$p_j = p_j(q,Q,t) = \frac{\partial F_1(q,Q,t)}{\partial q_j}$$

$$P_j = P_j(q,Q,t) = -\frac{\partial F_1(q,Q,t)}{\partial Q_j}$$

$F_1(q,Q,t)$ can then be obtained by directly integrating the above equations and combining the resulting expressions.

Note: Taking the respective partials of q and Q of the above equations,

$$-\frac{\partial P_j}{\partial q_i} = -\frac{\partial}{\partial q_i}\left(-\frac{\partial F_1(q,Q,t)}{\partial Q_j}\right) \qquad \frac{\partial}{\partial Q_j}\left(\frac{\partial F_1(q,Q,t)}{\partial q_j}\right) = \frac{\partial p_i}{\partial Q_j}$$

Canonical Transformation (more)

Now, from the pair of equations for the Generating Function Derivatives (Table 9.1), we form the following diff eqs,

$$p_j = p_j(q,Q,t) = \frac{\partial F_1\left(q,Q,t\right)}{\partial q_j}$$

$$P_j = P_j(q,Q,t) = -\frac{\partial F_1\left(q,Q,t\right)}{\partial Q_j}$$

$F_1(q,Q,t)$ can then be obtained by directly integrating the above equations and combining the resulting expressions.

Note: Since dF_1 is an exact differential wrt q and Q, so the two exps are equal,

$$-\frac{\partial P_j}{\partial q_i} = \frac{\partial^2 F_1}{\partial q_i \partial Q_j} = \frac{\partial^2 F_1}{\partial Q_j \partial q_i} = \frac{\partial p_i}{\partial Q_j}$$

(We will give the full list of relations later.)

Canonical Transformation (more)

Example (G8.2): We are given the following canonical transformation for a system with 1 dof:

$$Q = Q(q, p) = q\cos\alpha - p\sin\alpha$$
$$P = P(q, p) = q\sin\alpha + p\cos\alpha$$

(HW: showing this trans. is canonical)

(Q and P is being rotated in phase space from q and p by an angle α)

We seek a generating function of the 1st kind: $F_1(q, Q)$

First, notice that the cross-second derivatives for F_1 are equal as required for a canonical transformation:

$$\frac{\partial}{\partial Q}\left(\frac{\partial F_1}{\partial q}\right)\left[=\frac{\partial}{\partial Q}p\right]=\frac{\partial}{\partial Q}\left(-\frac{Q}{\sin\alpha}+q\cot\alpha\right)=-\frac{1}{\sin\alpha}$$

$$\frac{\partial}{\partial q}\left(\frac{\partial F_1}{\partial Q}\right)\left[=\frac{\partial}{\partial q}P\right]=\frac{\partial}{\partial q}\left(-\frac{q}{\sin\alpha}+Q\cot\alpha\right)=-\frac{1}{\sin\alpha}$$

Canonical Transformation (more)

Rewrite the transformation in terms of q and Q (indep. vars of F_1):

$$\frac{\partial F_1}{\partial q} = p = p(q,Q)$$
$$= -\frac{Q}{\sin\alpha} + q\cot\alpha$$

$$-\frac{\partial F_1}{\partial Q} = P = P(q,Q)$$
$$= q\sin\alpha + \left(-\frac{Q}{\sin\alpha} + q\frac{\cos\alpha}{\sin\alpha}\right)\cos\alpha$$
$$= \frac{q}{\sin\alpha} - Q\cot\alpha$$

Now, integrating the two partial differential equations:

$$F_1 = -\frac{Qq}{\sin\alpha} + \frac{q^2}{2}\cot\alpha + h(Q)$$

$$F_1 = -\frac{qQ}{\sin\alpha} + \frac{Q^2}{2}\cot\alpha + g(q)$$

Comparing these two expression, one possible solution for F_1 is,

$$\boxed{F_1(q,Q) = -\frac{Qq}{\sin\alpha} + \frac{1}{2}\left(q^2 + Q^2\right)\cot\alpha}$$

Canonical Transformation (more)

Now, let say we want to fine a Type-2 Generating function $F_2(q, P, t)$ for this problem...

As we have discussed previously, we can directly use the fact that F_2 is the Legendre transform of F_1,

$$F_1 = F_2(q, P, t) - Q_j P_j$$

$$F_2(q, P, t) = F_1(q, Q, t) + QP$$

$$F_2(q, P) = -\frac{Qq}{\sin\alpha} + \frac{1}{2}\left(q^2 + Q^2\right)\cot\alpha + QP$$

Now, from the CT, we can write Q by q and P (F_2 should be in q & P):

$$Q = q\cos\alpha - p\sin\alpha$$
$$P = q\sin\alpha + p\cos\alpha$$

$$Q = \frac{q}{\cos\alpha} - P\tan\alpha$$

Canonical Transformation (more)

As we have discussed previously, we can directly use the fact that F_2 is the Legendre transform of F_1,

$$F_1 = F_2(q, P, t) - Q_j P_j$$

$$F_2(q, P, t) = F_1(q, Q, t) + QP$$

$$F_2(q, P) = -\frac{Qq}{\sin\alpha} + \frac{1}{2}\left(q^2 + Q^2\right)\cot\alpha + QP$$

This then gives:

$$F_2(q, P) = \left(P - \frac{q}{\sin\alpha}\right)\underbrace{\left(\frac{q}{\cos\alpha} - P\tan\alpha\right)}_{Q} + \frac{1}{2}\left(q^2 + \left(\frac{q}{\cos\alpha} - P\tan\alpha\right)^2\right)\cot\alpha$$

Canonical Transformation (more)

As we have discussed previously, we can directly use the fact that F_2 is the Legendre transform of F_1,

$$F_1 = F_2(q,P,t) - Q_j P_j$$

$$F_2(q,P) = \left(P - \frac{q}{\sin\alpha}\right)\left(\frac{q}{\cos\alpha} - P\tan\alpha\right) + \frac{1}{2}\left(q^2 + \left(\frac{q}{\cos\alpha} - P\tan\alpha\right)^2\right)\cot\alpha$$

$$\frac{2qP}{\cos\alpha} - \frac{q^2}{\sin\alpha\cos\alpha} - P^2\tan\alpha$$

$$\frac{1}{2}\left(q^2\cot\alpha + \left(\frac{q^2}{\cos\alpha\sin\alpha} - \frac{2qP}{\cos\alpha} + P^2\tan\alpha\right)\right)$$

Canonical Transformation (more)

As we have discussed previously, we can directly use the fact that F_2 is the Legendre transform of F_1,

$$F_1 = F_2(q, P, t) - Q_j P_j$$

$$F_2\left(q,P\right) = \left(P - \frac{q}{\sin\alpha}\right)\left(\frac{q}{\cos\alpha} - P\tan\alpha\right) + \frac{1}{2}\left(q^2 + \left(\frac{q}{\cos\alpha} - P\tan\alpha\right)^2\right)\cot\alpha$$

$$\frac{\cancel{2}qP}{\cos\alpha} - \cancel{\frac{q^2}{\sin\alpha\cos\alpha}} - P^2\tan\alpha$$

$$\frac{1}{2}\left(q^2\cot\alpha + \left(\cancel{\frac{q^2}{\cos\alpha\sin\alpha}} - \cancel{\frac{2qP}{\cos\alpha}} + P^2\tan\alpha\right)\right)$$

Canonical Transformation (more)

As we have discussed previously, we can directly use the fact that F_2 is the Legendre transform of F_1,

$$F_1 = F_2(q,P,t) - Q_j P_j$$

$$F_2(q,P) = \left(P - \frac{q}{\sin\alpha}\right)\left(\frac{q}{\cos\alpha} - P\tan\alpha\right) + \frac{1}{2}\left(q^2 + \left(\frac{q}{\cos\alpha} - P\tan\alpha\right)^2\right)\cot\alpha$$

Finally, $$F_2(q,P) = \frac{qP}{\cos\alpha} - \frac{1}{2}\left(q^2 + P^2\right)\tan\alpha$$

Canonical Transformation (more)

Alternatively, we can substitute F into Eq. (*)G9.11 , results in replacing the $P_j\dot{Q}_j$ term by $-Q_j\dot{P}_j$ in our condition for a canonical transformation,

$$F = F_2(q,P,t) - Q_j P_j$$

$$p_j \dot{q}_j - H = -Q_j \dot{P}_j - K + \frac{dF}{dt}$$

Recall, this procedure gives us the two partial derivatives relations for F_2:

$$p_j = \frac{\partial F_2(q,P,t)}{\partial q_j} \qquad Q_j = \frac{\partial F_2(q,P,t)}{\partial P_j}$$

[Or, use the Table]

Canonical Transformation (more)

$$Q = Q(q,p) = q\cos\alpha - p\sin\alpha$$
$$P = P(q,p) = q\sin\alpha + p\cos\alpha$$

To solve for $F_2(q,P,t)$ in our example, again, we rewrite our given canonical transformation in q and P explicitly.

$$\frac{\partial F_2}{\partial q} = p = p(q,P) = \frac{P}{\cos\alpha} - q\tan\alpha$$

$$\frac{\partial F_2}{\partial P} = Q = Q(q,P) = q\cos\alpha - \left(\frac{P}{\cos\alpha} - q\frac{\sin\alpha}{\cos\alpha}\right)\sin\alpha = \frac{q}{\cos\alpha} - P\tan\alpha$$

Integrating and combining give,

$$\boxed{F_2(q,P) = \frac{qP}{\cos\alpha} - \frac{1}{2}\left(q^2 + P^2\right)\tan\alpha}$$

Canonical Transformation (more)

$$Q = Q(q,p) = q\cos\alpha - p\sin\alpha$$
$$P = P(q,p) = q\sin\alpha + p\cos\alpha$$

Notice that when $\alpha = 0$, $\sin\alpha = 0$

so that our coordinate transformation is just the identity transformation: $Q = q$ and $P = p$

⇨ *p, P* CANNOT be written explicitly in terms of q and Q !

⇨ so our assumption for using the type 1 generating function (with q and Q as indp var) cannot be fulfilled.

Consequently, $F_1(q,Q)$ blow up and cannot be used to derive the canonical transformation:

$$F_1(q,Q) = -\frac{Qq}{\sin\alpha} + \frac{1}{2}\left(q^2 + Q^2\right)\cot\alpha \quad \to \infty \quad \text{as } \alpha \to 0$$

But, using a Type 2 generating function will work.

Canonical Transformation (more)

$$Q = Q(q,p) = q\cos\alpha - p\sin\alpha$$
$$P = P(q,p) = q\sin\alpha + p\cos\alpha$$

Similarly, we can see that when $\alpha = \dfrac{\pi}{2}$, $\cos\alpha = 0$

our coordinate transformation is a coordinate switch $Q = -p$, $P = q$

➡ *p, Q* CANNOT be written explicitly in terms of *q* and *P* !

➡ so the assumption for using the type 2 generating function (with *q* and *P* as indp var) cannot be fulfilled.

Consequently, $F_2(q,P)$ blow up and cannot be used to derive the canonical transformation:

$$F_2(q,P) = \frac{qP}{\cos\alpha} - \frac{1}{2}\left(q^2 + P^2\right)\tan\alpha \quad \to \infty \quad \text{as } \alpha \to 0$$

But, using a Type 1 generating function will work in this case.

Canonical Transformation (more)

- A suitable generating function doesn't have to conform to only one of the four types for *all* the degrees of freedom in a given problem !
- There can also be more than one solution for a given CT
- First , we need to choose a suitable set of *independent* variables for the generating function.

→For a generating function to be useful, it should depends on half of the old and half of the new variables

→As we have done in the previous example, the procedure in solving for F involves integrating the partial derivative relations resulted from "consistence" considerations using the main condition for a canonical transformation, i.e.,

$$p_j \dot{q}_j - H(q, p, t) = P_j \dot{Q}_j - K(Q, P, t) + \frac{dF}{dt} \quad (G9.11)$$

Canonical Transformation (more)

→For these partial derivative relations to be solvable, one must be able to feed-in 2*n* *independent* coordinate relations (from the given CT) in terms of a chosen set of ½ new + ½ old variables.

- In general, one can use ANY one of the four types of generating functions for the canonical transformation as long as the RHS of the transformation can be written in terms of the associated pairs of phase space coordinates: (q, Q, t), (q, P, t), (q, Q, t), or (p, P, t).

$$\quad T1 \qquad T2 \qquad T3 \qquad T4$$

- On the other hand, if the transformation is such that the RHS cannot written in term of a particular pair: (q, Q, t), (q, P, t), (q, Q, t), or (p, P, t), then that associated type of generating functions cannot be used.

Canonical Transformation: an example with two dofs

- To see in practice how this might work… Let say, we have the following transformation involving 2 dofs: $\left(q_1, p_1, q_2, p_2\right) \rightarrow \left(Q_1, P_1, Q_2, P_2\right)$

$$Q_1 = q_1 \quad (1a) \qquad Q_2 = p_2 \quad (2a)$$
$$P_1 = p_1 \quad (1b) \qquad P_2 = -q_2 \quad (2b)$$

- As we will see, this will involve a *mixture* of two different basic types.

Canonical Transformation: an example with two dofs

$$Q_1 = q_1 \quad (1a) \qquad\qquad Q_2 = p_2 \quad (2a)$$
$$P_1 = p_1 \quad (1b) \qquad\qquad P_2 = -q_2 \quad (2b)$$

- First, let see if we can use the simplest type (type 1) for both dofs, i.e., *F* will depend only on the *q*-*Q*'s:

$$F(q_1, q_2, Q_1, Q_2, t)$$

→Notice that Eq (1a) is a relation linking q_1, Q_1 only , they CANNOT both be independent variables → Type 1 (only) WON'T work !

- As an alternative, we can try to use the set p_1, q_2, Q_1, Q_2 as our independent variables. This will give an *F* which is a mixture of Type 3 and 1.

(In Goldstein (p. 377), another alternative was using q_1, q_2, P_1, Q_2 resulted in a different generating function which is a mixture of Type 2 and 1.)

Canonical Transformation: an example with two dofs

From our CT, we can write down the following relations:

Dependent variables q_1, p_2, P_1, P_2

Independent variables p_1, q_2, Q_1, Q_2

$$\left[\begin{array}{l} q_1 = Q_1 \\ p_2 = Q_2 \\ P_1 = p_1 \\ P_2 = -q_2 \end{array}\right] \qquad (*)$$

Now, with this set of ½ new + ½ old independent variable chosen, we need to derive the set of partial derivative conditions by substituting $F(p_1, q_2, Q_1, Q_2, t)$ into Eq. 9.11 (or look them up from the Table).

Canonical Transformation: an example with two dofs

The explicit independent variables (those appear in the differentials) in Eq. 9.11 are the q-Q's. To do the conversion:

$$q_1, q_2, Q_1, Q_2 \quad \text{(Eq. 9.11's explicit ind vars)}$$

$$\updownarrow$$

$$p_1, q_2, Q_1, Q_2 \quad \text{(our preferred ind vars)}$$

we will use the following Legendre transformation: $F = F'(p_1, q_2, Q_1, Q_2, t) + q_1 p_1$

Substituting this into Eq. 9.11, we have:

$$\cancel{p_1\dot{q}_1} + p_2\dot{q}_2 - H = P_1\dot{Q}_1 + P_2\dot{Q}_2 - K + \frac{dF}{dt}$$

$$= P_1\dot{Q}_1 + P_2\dot{Q}_2 - K + \frac{\partial F'}{\partial p_1}\dot{p}_1 + \frac{\partial F'}{\partial q_2}\dot{q}_2 + \frac{\partial F'}{\partial Q_1}\dot{Q}_1 + \frac{\partial F'}{\partial Q_2}\dot{Q}_2 + q_1\dot{p}_1 + \cancel{p_1\dot{q}_1} + \frac{\partial F'}{\partial t}$$

Canonical Transformation: an example with two dofs

$$p_2\dot{q}_2 - H = P_1\dot{Q}_1 + P_2\dot{Q}_2 - K + \frac{\partial F'}{\partial p_1}\dot{p}_1 + \frac{\partial F'}{\partial q_2}\dot{q}_2 + \frac{\partial F'}{\partial Q_1}\dot{Q}_1 + \frac{\partial F'}{\partial Q_2}\dot{Q}_2 + q_1\dot{p}_1 + \frac{\partial F'}{\partial t}$$

Comparing terms, we have the following conditions:

$$q_1 = -\frac{\partial F'}{\partial p_1} \qquad P_1 = -\frac{\partial F'}{\partial Q_1}$$

$$p_2 = \frac{\partial F'}{\partial q_2} \qquad P_2 = -\frac{\partial F'}{\partial Q_2}$$

$$K = H + \frac{\partial F'}{\partial t}$$

As advertised, this is a mixture of Type 3 and 1 of the basic CT.

Canonical Transformation: an example with two dofs

$$q_1 = Q_1$$
$$p_2 = Q_2$$
$$P_1 = p_1$$
$$P_2 = -q_2$$

Substituting our coordinates transformation [Eq. (*)] into the partial derivative relations, we have :

$$\frac{\partial F'}{\partial p_1} = -q_1 = -Q_1 \qquad F' = -Q_1 p_1 + f(q_2, Q_1, Q_2)$$

$$\frac{\partial F'}{\partial Q_1} = -P_1 = -p_1 \qquad F' = -p_1 Q_1 + g(p_1, q_2, Q_2)$$

$$\frac{\partial F'}{\partial Q_2} = -P_2 = -(-q_2) \qquad F' = q_2 Q_2 + h(p_1, q_2, Q_1)$$

$$\frac{\partial F'}{\partial q_2} = p_2 = Q_2 \qquad F' = Q_2 q_2 + k(p_1, Q_1, Q_2)$$

$$\boxed{F' = -p_1 Q_1 + q_2 Q_2}$$

(Note: Choosing q_1, q_2, P_1, Q_2 instead, Goldstein has $F'' = q_1 P_1 + q_2 Q_2$. Both of these are valid generating functions.)

$$Q_j = Q_j(q,p,t)$$
$$P_j = P_j(q,p,t)$$

Canonical Transformation: Review

$$p_j\dot{q}_j - H(q,p,t) = P_j\dot{Q}_j - K(Q,P,t) + \frac{dF}{dt}(old, new, t)$$

Type 1: $F = F_1(q,Q,t)$	$p_j = \frac{\partial F_1}{\partial q_j}(q,Q,t) \quad P_j = -\frac{\partial F_1}{\partial Q_j}(q,Q,t)$	$K = H + \frac{\partial F_1}{\partial t}$
Type 2: $F = F_2(q,P,t) - Q_i P_i$	$p_j = \frac{\partial F_2}{\partial q_j}(q,P,t) \quad Q_j = \frac{\partial F_2}{\partial P_j}(q,P,t)$	$K = H + \frac{\partial F_2}{\partial t}$
Type 3: $F = F_3(p,Q,t) + q_i p_i$	$q_j = -\frac{\partial F_3}{\partial p_j}(p,Q,t) \quad P_j = -\frac{\partial F_3}{\partial Q_j}(p,Q,t)$	$K = H + \frac{\partial F_3}{\partial t}$
Type 4: $F = F_4(p,P,t) + q_i p_i - Q_i P_i$	$q_j = -\frac{\partial F_4}{\partial p_j}(p,P,t) \quad Q_j = \frac{\partial F_4}{\partial P_j}(p,P,t)$	$K = H + \frac{\partial F_4}{\partial t}$

Canonical Transformation: Review

- **Generating function** is useful as a bridge to link half of the original set of coordinates (either q or p) to another half of the new set (either Q or P).

- In general, one can use ANY one of the four types of generating functions for the canonical transformation as long as the transformation can be written in terms of the associated pairs of phase space coordinates: (q, Q, t), (q, P, t), (q, Q, t), or (p, P, t).

- On the other hand, if the transformation is such that it cannot be written in term of a particular pair: (q, Q, t), (q, P, t), (q, Q, t), or (p, P, t), then that associated type of generating functions cannot be used.

- the procedure in solving for F involves integrating the resulting partial derivative relations from the CT condition